Supervised Readings I

Linear Representations of Finite Groups

Artin's Theorem

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Preliminary

Prior to stating Artin's theorem, we shall define the notion on 'characters' and 'rings'.

Definition 1 : character

From $\rho : G \longrightarrow GL(V)$ be a linear representation of finite group G to the vector space V, we take $\chi_\rho(s) = Tr(\rho_s)$ as the character of ρ, $\forall s \in G$.

Proposition 1.1 : if ρ has degree $n \Rightarrow$ (i) $\chi(1) = n$ since we have $\dim(V) = n$, $\rho(1) = 1$ and $Tr(1) = n$. (ii) $\chi(s^{-1}) = \overline{\chi(s)}$, $\forall s \in G$. We set λ_i be the eigenvalue, then $\chi(\bar{s}) = Tr(\overline{\rho_s}) = \sum_i \overline{\lambda_i} = \sum_i \lambda_i^{-1} = Tr(\rho_s^{-1}) = Tr(\rho_{s^{-1}}) = \chi(s^{-1})$

(iii) $\chi(t^{-1}st) = \chi(s)$, $\forall t, s \in G$. Set $u = ts$, $v = t^{-1} \Rightarrow Tr(uv) = Tr(vu)$

By (iii), let us recall class functions : a function f on G is called a 'class function' if $f(t^{-1}st) = f(s)$ $\forall s, t \in G$.

Definition 2 : Induced representation and characters

Let $H \leq G$ be a subgroup ; R be a system of left cosets for H ; $V = \mathbb{C}[G]$ - module ; $W =$ sub-$\mathbb{C}[G]$- module of V, then V is induced by W if $V = \bigoplus_{s \in R} sW$. Here, we may reform $W' = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$.

i.e. the induced representation $V = \mathrm{Ind}_H^G W$ in which can be gotten from W by scalar extension from $\mathbb{C}[H]$-module to $\mathbb{C}[G]$.

Remark : Every representation of G defines a unique left $\mathbb{C}[G]$-module where $\mathbb{C}[G]$ is a group ring of formal sums of elements of G with complex coefficients.

i.e. $\mathbb{C}[G]$ is the algebra G over $\mathbb{C}$, for each $f \in \mathbb{C}[G]$ can be uniquely written as $f = \sum_{s \in G} a_s s$ $\forall a_s \in \mathbb{C}$.

Further on class function : let f be a class function on H, if we have $f' = \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}st \in H}} f(t^{-1}st)$, then $f' = \mathrm{Ind}_H^G(f)$ i.e. f' is induced by f.

Proposition 2.1 : (i) f' is a class function on G, (ii) If f is a character of W of $H \leq G$, $\mathrm{Ind}_H^G(f)$ is also a character of $\mathrm{Ind}_H^G(W)$ of G.

Observe that each class function is a linear combination of characters, if φ_1, φ_2 be class functions on G, then $\langle \varphi_1, \varphi_2 \rangle = \frac{1}{|G|} \sum_{s \in G} \varphi_1(s^{-1}) \varphi_2(s)$, $\forall s \in G$.

If V_1, V_2 be two $\mathbb{C}[G]$-modules, then $\langle V_1, V_2 \rangle_G = \dim \mathrm{Hom}^G(V_1, V_2)$.

Lemma 2.2 : Using prop 2.1, $\langle \varphi_1, \varphi_2 \rangle_G = \langle V_1, V_2 \rangle_G$ by orthogonality.

Theorem 2.3 : Let ψ be a class function on H, φ be a class function on G, then

$\langle \psi, \mathrm{Res}\, \varphi \rangle_H = \langle \mathrm{Ind}\, \varphi, \psi \rangle_G$, in fact, this gives us

to the formula $\mathrm{Ind}(\psi \cdot \mathrm{Res}\, \varphi) = (\mathrm{Ind}\, \psi) \cdot \varphi$ ..... $(*)$

Definition 3 : the ring $R(G)$

From a finite group G, let $\chi_1, \ldots, \chi_h$ be distinct irreducible characters.

Then a class function F on G is a character if and only if $F = \sum n_i \chi_i$ for $n_i \in \mathbb{Z}^+$

Let us denote $R^+(G) = \{ F(t^{-1}st) = F(s)\ \forall t, s \in G \mid F = \sum n_i \chi_i \text{ for } n_i \in \mathbb{Z}^+ \}$, and

$R(G) = \mathbb{Z}\chi_1 \oplus + \cdots \oplus \mathbb{Z}\chi_h$, the group generated by $R^+(G)$. Here the element $k \in R(G)$ is called a 'virtual character', and $R(G) \subset F_{\mathbb{C}}(G)$ is a subring.

Proposition 3.1 : If $H \leq G$, then (i) $\mathrm{Res} : R(G) \longrightarrow R(H)$, (ii) $\mathrm{Ind} : R(H) \to R(G)$

is a ring homomorphism by Frobenius reciprocity.

Furthermore, by $(*)$, the image of $\mathrm{Ind} : R(H) \longrightarrow R(G)$ is an ideal of $R(G)$.

Proposition 3.2: If we have commutative ring A, (i) and (ii) from prop 3.1 extended by linearity to A-linear maps:

$$\left.\begin{array}{l} \text{(i)}' \ A \otimes \mathrm{Res} : A \otimes R(G) \longrightarrow A \otimes R(H) \\ \text{(ii)}'' \ A \otimes \mathrm{Ind} : A \otimes R(H) \longrightarrow A \otimes R(G) \end{array}\right\} \cdots\cdots (**)$$

We may use $(**)$ for Artin's theorem.

Artin's Theorem

Theorem 4. Let X be a family of $H_i \leq G$ subgroups; $\mathrm{Ind} : \bigoplus_{H \in X} R(H) \longrightarrow R(G)$ be the homomorphism, then (i) G is the union of the conjugates of the subgroups belonging to X and (ii) the cokernel of $\mathrm{Ind} : \bigoplus_{H \in X} R(H) \longrightarrow R(G)$ is finite.

By def 3, since $R(G)$ is finitely generated as a group, we may rephrase as (ii)' for each character χ of G, there exist virtual charaters $\chi_H \in R(H)$ where $H \in X$, and an integer $d \geq 1$, such that $d\chi = \sum_{H \in X} \mathrm{Ind}_H^G(\chi_H)$.

Proof 4.1. We want to show (ii) $\Longrightarrow$ (i)

Suppose (ii) is satisfied. We first see that $\mathrm{coker}(\mathrm{Ind}) = R(G)/\mathrm{Image}(\mathrm{Ind})$ is finite so $|\mathrm{coker}(\mathrm{Ind})| < \infty$. Let S be the union of conjugates of the subgroups $H \in X$. i.e. $S = \bigcup_{\substack{s \in G \\ H \in X}} sHs^{-1}$. Then for any $f_H \in R(H)$, each function of the form $\sum \mathrm{Ind}_H^G(f_H)$ vanishes off S. Indeed by (ii)' where $R(G) = \bigoplus_{i=1}^{h} \mathbb{Z}\chi_i$, a virtual character, and for each $\chi \in R(G)$ we have

$$\chi = \sum_{H \in X} \frac{1}{d} \mathrm{Ind}_H^G(\chi_H) = \frac{1}{d} \sum_{H \in X} \mathrm{Ind}_H^G(\chi_H) = \frac{1}{|R(G)/\mathrm{Image}(\mathrm{Ind})|} \mathrm{Ind}_H^G(\chi_H)$$

such that virtual characters $\chi_H \in R(H)$ exists. Since $d \geq 1$ is an integer, any function of the form $\sum \mathrm{Ind}_H^G(\chi_H)$ vanishes off on S ; (ii) is already satisfied each class function on G also vanishes off on S; hence, we find

$S = \bigcup_{\substack{H \in X \\ s \in G}} sHs^{-1} = G$, and this clearly implies (i) $\square$

Proof 4.2. We want to show (i) $\Longrightarrow$ (ii) .

Prior to proceed, we may use (**) from proposition 3.2. Suppose (i) is satisfied, then $\mathbb{Q} \otimes \mathrm{Ind} : \bigoplus_{H \in X} \mathbb{Q} \otimes R(H) \longrightarrow \mathbb{Q} \otimes R(G)$ is surjective.

This is equivalent to $\mathbb{C} \otimes \mathrm{Ind} : \bigoplus_{H \in X} \mathbb{C} \otimes R(H) \longrightarrow \mathbb{C} \otimes R(G)$; hence, by duality, $\mathbb{C} \otimes \mathrm{Res} : \mathbb{C} \otimes R(G) \longrightarrow \bigoplus_{H \in X} \mathbb{C} \otimes R(H)$ is injective .

So, if $f \in G$ is a class function in which restricts to 0 on each cyclic subgroup it is clearly zero , the cokernel is finite.

Let $A \leq G$ is cyclic, and satisfies (i), then each $\chi \in G$ is a linear combination with coefficients $\mathbb{Q}$ in which induced by χ_A of A.

We define propositions below to prove the theorem.

Proposition 4.2.1. Let us construct a function θ_A where $A \leq G$ cyclic with order a : $\theta_A = \begin{cases} a, & \text{if } x \text{ generates } A \\ 0, & \text{otherwise} \end{cases}$. Now if $|G| = g$ finite, then $g = \sum_{A \leq G} \mathrm{Ind}_A^G(\theta_A)$.

Note that g is a constant function that equals to the order of G. We take

$\theta'_A = \frac{1}{a} \sum \theta_A(yxy^{-1})$ for each $x, y \in G$, $yxy^{-1} \in A$.

$= \frac{1}{a} \sum a = \sum 1$ since $y \in G$ and yxy^{-1} generates A.

Since this is unique, $\sum_{A \leq G} \theta'_A(x) = \sum_{y \in G} 1 = g$.

Proposition 4.2.2. Suppose $A \leq G$ is cyclic, then $\theta_A \in R(A)$.

By previous proposition, $\sum_{B \leq A} \mathrm{Ind}_B^A(\theta_B) = \theta_A + \sum_{B \neq A} \mathrm{Ind}_B^A(\theta_B) = a$.

Here, by strong induction, for $B \neq A$, we have $\theta_B \in R(B)$ so that $\mathrm{Ind}_B^A(\theta_B) \in R(A)$

Conversely, for $a \in R(A)$ we also see that $\theta_A \in R(A)$.

We then use prop 4.2.1. and 4.2.2. to show (i) $\Rightarrow$ (ii).

Now, consider $A' \subset yAy^{-1}$; i.e. the conjugate of A, then we observe from

$\mathrm{Ind}: R(A') \longrightarrow R(G)$ to $\mathrm{Ind}: R(A) \longrightarrow R(G)$, the image of $\mathrm{Ind}_{A'}^G$ is

clearly contained in Image (Ind_A^G). So, $X = \bigcup_{A \in G} \theta_A$, and by previous

propositions, we have seen that $g = \sum_{A \in X} \mathrm{Ind}_A^G(\theta_A)$ where $\theta_A \in R(A)$,

$g \in \text{Image}(\mathrm{Ind}_A^G)$. In fact, by proposition 3.1, Image (Ind_A^G) is an

ideal of $R(G)$. Hence, for each $\chi \in R(G)$, this image (Ind_A^G) contains

every element of the form $g\chi$. i.e. we have $g \geq 1$ be positive integer,

there exist some virtual characters $\chi_A \in R(A)$, $A \leq G$, such that

$$g\chi = \sum_{A \in X} \mathrm{Ind}_A^G(\chi_A)$$

. This implies (ii)' $\square$